Lecture 17

Dielectric Waveguides

As mentioned before, the dielectric waveguide shares many salient features with the optical fiber waveguide, one of the most important waveguides of this century. Before we embark on the study of dielectric waveguides, we will revisit the transverse resonance again. The transverse resonance condition allows one to derive the guidance conditions for a dielectric waveguide easily without having to match the boundary conditions at the interface again: The boundary conditions are already used when deriving the Fresnel reflection coefficients, and hence they are embedded in these reflection coefficients and generalized reflection coefficients. Much of the materials in this lecture can be found in [32, 44, 83].

17.1 Generalized Transverse Resonance Condition

The guidance conditions, the transverse resonance condition given previously, can also be derived for the more general case. The generalized transverse resonance condition is a powerful condition that can be used to derive the guidance condition of a mode in a layered medium.

To derive this condition, we first have to realize that a guided mode in a waveguide is due to the coherent or constructive interference of the waves. This implies that if a plane wave starts at position 1 (see Figure 17.1)[1] and is multiply reflected as shown, it will regain its original phase in the x direction at position 5. Since this mode progresses in the z direction, all these waves (also known as partial waves) are in phase in the z direction by the phase matching condition. Otherwise, the boundary conditions cannot be satisfied. That is, waves at 1 and 5 will gain the same phase in the z direction. But, for them to add coherently or interfere coherently in the x direction, the transverse phase at 5 must be the same as 1.

Assuming that the wave starts with amplitude 1 at position 1, it will gain a transverse phase of $e^{-j\beta_{0x}t}$ when it reaches position 2. Upon reflection at $x = x_2$, at position 3, the wave becomes $\tilde{R}_+ e^{-j\beta_{0x}t}$ where $\tilde{R}_+$ is the generalized reflection coefficient at the right interface of Region 0. Finally, at position 5, it becomes $\tilde{R}_- \tilde{R}_+ e^{-2j\beta_{0x}t}$ where $\tilde{R}_-$ is the generalized

[1]The waveguide convention is to assume the direction of propagation to be z. Since we are analyzing a guided mode in a layered medium, z axis is as shown in this figure, which is parallel to the interfaces. This is different from before.

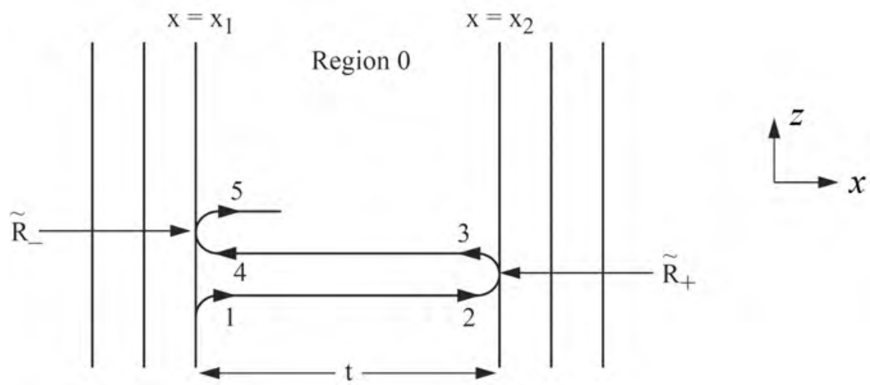

Figure 17.1: The transverse resonance condition for a layered medium. The phase of the wave at position 5 should be equal to the transverse phase at position 1 for constructive interference to occur.

reflection coefficient at the left interface of Region 0. For constructive interference to occur or for the mode to exist, we require that

$$\tilde{R}_{-}\tilde{R}_{+}e^{-2j\beta_{0x}t} = 1 \tag{17.1.1}$$

The above is the generalized transverse resonance condition for the guidance condition for a plane wave mode traveling in a layered medium.

In (17.1.1), a metallic wall has a reflection coefficient of 1 for a TM wave; hence if $\tilde{R}_{+}$ is 1, Equation (17.1.1) becomes

$$1 - \tilde{R}_{-}e^{-2j\beta_{0x}t} = 0. \tag{17.1.2}$$

On the other hand, in (17.1.1), a metallic wall has a reflection coefficient of −1, for TE wave, and Equation (17.1.1) becomes

$$1 + \tilde{R}_{-}e^{-2j\beta_{0x}t} = 0. \tag{17.1.3}$$

17.2 Dielectric Waveguide

The most important dielectric waveguide of the modern world is the optical fiber, whose invention was credited to Charles Kao [98]. He was awarded the Nobel prize in 2009 [112]. However, the analysis of the optical fiber requires the use of cylindrical coordinates and special functions such as Bessel functions. In order to capture the essence of dielectric waveguides, one can study the slab dielectric waveguide, which shares many salient features with the optical fiber. This waveguide is also used as thin-film optical waveguides (see Figure 17.2). We start with analyzing the TE modes in this waveguide.

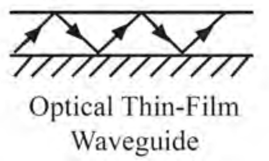

Figure 17.2: An optical thin-film waveguide is made by coating a thin dielectric film or sheet on a metallic surface. The wave is guided by total internal reflection at the top interface, and by metallic reflection at the bottom interface.

17.2.1 TE Case

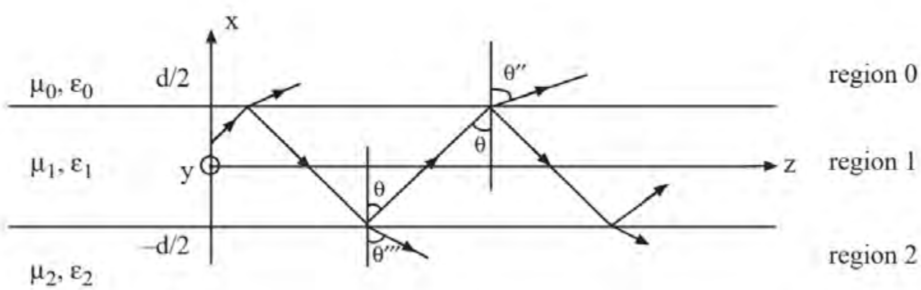

Figure 17.3: Schematic for the analysis of a guided mode in the dielectric waveguide. Total internal reflections occur at the top and bottom interfaces. If the waves add coherently, the wave is guided along the dielectric slab.

We shall look at the application of the transverse resonance condition to a TE wave guided in a dielectric waveguide. Again, we assume the direction of propagation of the guided mode to be in the z direciton in accordance with convention. Specializing the above equation to the dielectric waveguide shown in Figure 17.3, we have the guidance condition as

$$1 = R_{10} R_{12} e^{-2j\beta_{1x} d} \tag{17.2.1}$$

where d is the thickness of the dielectric slab. Guidance of a mode is due to total internal reflection, and hence, we expect Region 1 to be optically more dense (in terms of optical refractive indices)[2] than region 0 and 2.

To simplify the analysis further, we assume Region 2 to be the same as Region 0 so that $R_{12} = R_{10}$. The new guidance condition is then

$$1 = R_{10}^2 e^{-2j\beta_{1x} d} \tag{17.2.2}$$

[2]Optically more dense means higher optical refractive index, or higher dielectric constant.

By phase-matching, β_z is the same in all the three regions of Figure 17.3. By expressing all the β_{ix} in terms of the variable β_z, the above is an implicit equation for β_z. Also, we assume that $\varepsilon_1 > \varepsilon_0$ so that total internal reflection occurs at both interfaces as the wave bounces around so that $\beta_{0x} = -j\alpha_{0x}$. Therefore, for TE polarization, the local, single-interface, or Fresnel reflection coefficient is

$$R_{10} = \frac{\mu_0\beta_{1x} - \mu_1\beta_{0x}}{\mu_0\beta_{1x} + \mu_1\beta_{0x}} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = e^{j\theta_{TE}} \tag{17.2.3}$$

where θ_{TE} is the Goos-Hanschen shift for total internal reflection. It is given by

$$\theta_{TE} = 2\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right) \tag{17.2.4}$$

The guidance condition for constructive interference according to (17.2.1) is such that

$$2\theta_{TE} - 2\beta_{1x}d = 2n\pi \tag{17.2.5}$$

From the above, dividing it by four, and taking its tangent, we get

$$\tan\left(\frac{\theta_{TE}}{2}\right) = \tan\left(\frac{n\pi}{2} + \frac{\beta_{1x}d}{2}\right) \tag{17.2.6}$$

or using (17.2.4) for the left-hand side,

$$\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}} = \tan\left(\frac{n\pi}{2} + \frac{\beta_{1x}d}{2}\right) \tag{17.2.7}$$

The above gives rise to

$$\mu_1\alpha_{0x} = \mu_0\beta_{1x}\tan\left(\frac{\beta_{1x}d}{2}\right), \qquad n \text{ even} \tag{17.2.8}$$

$$-\mu_1\alpha_{0x} = \mu_0\beta_{1x}\cot\left(\frac{\beta_{1x}d}{2}\right), \qquad n \text{ odd} \tag{17.2.9}$$

It can be shown that when n is even, the mode profile is even, whereas when n is odd, the mode profile is odd. The above can also be rewritten as

$$\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2}, \qquad \text{even modes} \tag{17.2.10}$$

$$-\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\frac{\beta_{1x}d}{2}\right) = \frac{\alpha_{0x}d}{2}, \qquad \text{odd modes} \tag{17.2.11}$$

Again, the above equations can be expressed in the β_z variable, but they do not have closed form solutions, save for graphical solutions (or numerical solutions). We shall discuss their graphical solutions.[3]

[3]This technique has been put together by the community of scholars in the optical waveguide area.

To solve the above graphically, it is best to plot them in terms of one common variable. It turns out the β_{1x} is the simplest common variable to use for graphical solutions. To this end, using the fact that $-\alpha_{0x}^2 = \beta_0^2 - \beta_z^2$, and that $\beta_{1x}^2 = \beta_1^2 - \beta_z^2$, eliminating β_z from these two equations, one can show that

$$\alpha_{0x} = [\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0) - \beta_{1x}^2]^{\frac{1}{2}} \tag{17.2.12}$$

Thus (17.2.10) and (17.2.11) become

$$\begin{aligned}
\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) &= \frac{\alpha_{0x}d}{2} \\
&= \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2}, \quad \text{even modes} \qquad (17.2.13) \\
-\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\frac{\beta_{1x}d}{2}\right) &= \frac{\alpha_{0x}d}{2} \\
&= \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2}, \quad \text{odd modes} \qquad (17.2.14)
\end{aligned}$$

We can solve the above graphically by plotting

$$y_1 = \frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\tan\left(\frac{\beta_{1x}d}{2}\right) \quad \text{even modes} \tag{17.2.15}$$

$$y_2 = -\frac{\mu_0}{\mu_1}\frac{\beta_{1x}d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) \quad \text{odd modes} \tag{17.2.16}$$

$$y_3 = \left[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\frac{\beta_{1x}d}{2}\right)^2\right]^{\frac{1}{2}} = \frac{\alpha_{0x}d}{2} \tag{17.2.17}$$

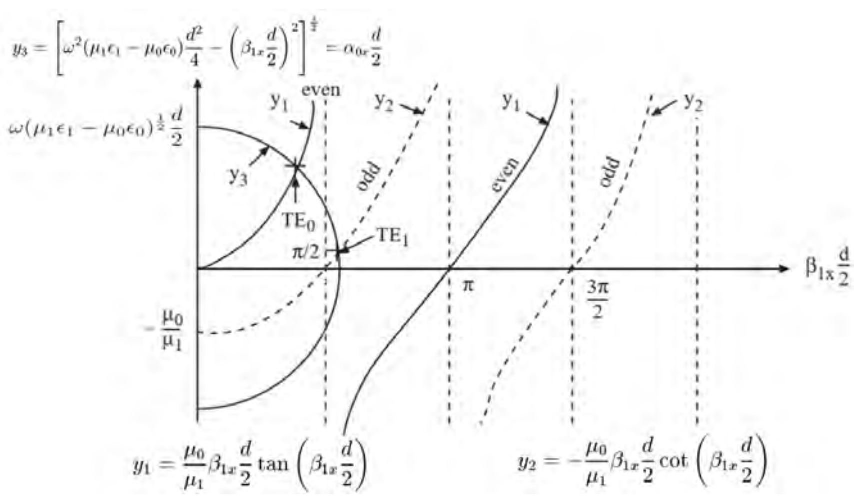

Figure 17.4: A way to solve (17.2.13) and (17.2.14) is via a graphical method. In this method, both the right-hand side and the left-hand side of the equations are plotted on the same plot. The solutions are at the intersection points of these plots.

In the above, y_3 is the equation of a circle; the radius of the circle is given by

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2}. \tag{17.2.18}$$

The solutions to (17.2.13) and (17.2.14) are given by the intersections of y_3 with y_1 and y_2. We note from (17.2.1) that the radius of the circle can be increased in three ways: (i) by increasing the frequency, (ii) by increasing the contrast $\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}$, and (iii) by increasing the thickness d of the slab.[4] The mode profiles of the first two modes are shown in Figure 17.5.

[4]These area important salient features of a dielectric waveguide. These features are also shared by the optical fiber.

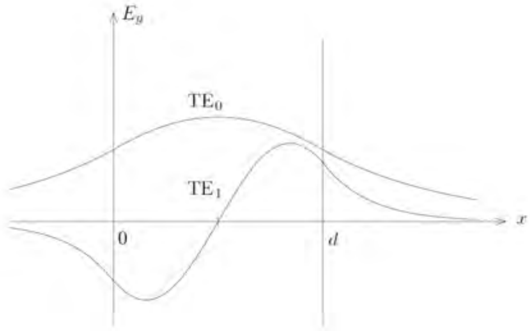

Figure 17.5: Mode profiles of the TE$_0$ and TE$_1$ modes of a dielectric slab waveguide (courtesy of J.A. Kong [32]).

When $\beta_{0x} = -j\alpha_{0x}$, the reflection coefficient for total internal reflection is

$$R_{10}^{TE} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = \exp\left[+2j\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right)\right] \tag{17.2.19}$$

and $\left|R_{10}^{TE}\right| = 1$. Hence, the wave is guided by total internal reflections.

Cut-off occurs when the total internal reflection ceases to occur, i.e. when the frequency decreases such that $\alpha_{0x} = 0$. From Figure 17.4, we see that $\alpha_{0x} = 0$ when

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2} = \frac{m\pi}{2}, \qquad m = 0, 1, 2, 3, \ldots \tag{17.2.20}$$

or

$$\omega_{mc} = \frac{m\pi}{d(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}}, \qquad m = 0, 1, 2, 3, \ldots \tag{17.2.21}$$

The mode that corresponds to the m-th cut-off frequency above is labeled the TE$_m$ mode. Thus TE$_0$ mode is the mode that has no cut-off or propagates at all frequencies. This is shown in Figure 17.6 where the TE mode profiles are similar since they are dual to each other. The boundary conditions at the dielectric interface is that the field and its normal derivative have to be continuous. The TE$_0$ or TM$_0$ mode can satisfy this boundary condition at all frequencies, but not the TE$_1$ or TM$_1$ mode. At the cut-off frequency, the field outside the slab has to become flat implying the $\alpha_{0x} = 0$ implying no guidance.

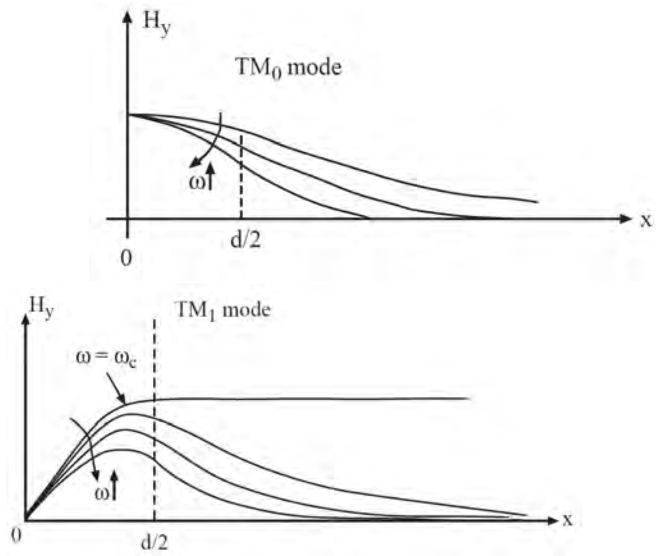

Figure 17.6: Mode profiles of the TM modes of a dielectric slab. The TE modes are dual to the TM modes and have similar mode profiles.

Next, we will elucidate more physics of the dielectric slab guided mode. At cut-off, $\alpha_{0x} = 0$, and from the dispersion relation that $\alpha_{0x}^2 = \beta_z^2 - \beta_0^2$,

$$\beta_z = \omega\sqrt{\mu_0\epsilon_0},$$

for all the modes. Hence, the phase velocity, ω/β_z, and the group velocity, $d\omega/d\beta_z$ are that of the outer region. This is because when $\alpha_{0x} = 0$, the wave is not evanescent outside, and the energy of the mode is predominantly carried by the exterior field.

When $\omega \to \infty$, the radius of the circle in the plot of y_3 becomes increasingly larger. As seen from Figure 17.4, the solution for $\beta_{1x} \to \frac{n\pi}{d}$ for all the modes. From the dispersion relation for Region 1,

$$\beta_z = \sqrt{\omega^2\mu_1\epsilon_1 - \beta_{1x}^2} \approx \omega\sqrt{\mu_1\epsilon_1}, \qquad \omega \to \infty \tag{17.2.22}$$

since $\omega^2\mu_1\epsilon_1 \gg \beta_{1x}^2 = (n\pi/d)^2$. Hence the group and phase velocities approach that of the dielectric slab. This is because when $\omega \to \infty$, $\alpha_{0x} \to \infty$, implying the rapid exponential decay of the fields outside the waveguide. Therefore, the fields are trapped or confined in the slab and propagating within it. Because of this, the dispersion diagram of the different modes appear as shown in Figure 17.7. In this figure,[5] k_{c1}, k_{c2}, and k_{c3} are the cut-off wave number or frequency of the first three modes. Close to cut-off, the field is traveling mostly outside the waveguide, and $k_z \approx \omega\sqrt{\mu_0\varepsilon_0}$. Hence, both the phase and group velocities approach that of

[5]Please note again that in this course, we will use β and k interchangeably for wavenumbers.

the outer medium as shown in the figure. When the frequency increases, the mode is tightly confined in the dielectric slab, and hence, $k_z \approx \omega\sqrt{\mu_1\varepsilon_1}$. Both the phase and group velocities approach that of Region 1 as shown.

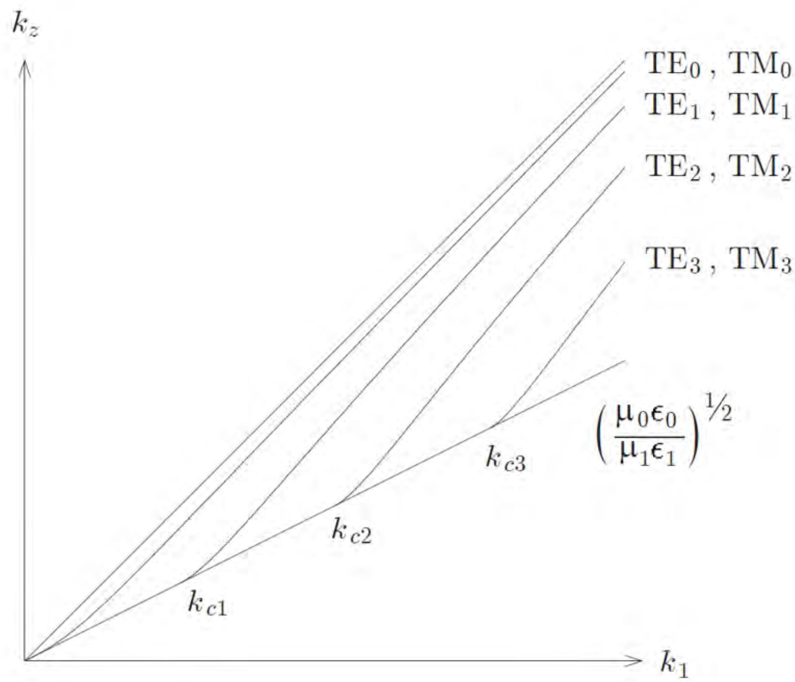

Figure 17.7: Here, we have k_z versus k_1 plots for dielectric slab waveguide. Near its cut-off, the energy of the mode is in the outer region, and hence, its group velocity is close to that of the outer region. At high frequencies, the mode is tightly bound to the slab, and its group velocity approaches that of the dielectric slab (courtesy of J.A. Kong [32]).

17.2.2 TM Case

For the TM case, a similar guidance condition analogous to (17.2.1) can be derived but with the understanding that the reflection coefficients in (17.2.1) are now TM reflection coefficients. Similar derivations show that the above guidance conditions, for $\epsilon_2 = \epsilon_0$, $\mu_2 = \mu_0$, reduce to

$$\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2}, \qquad \text{even modes} \qquad (17.2.23)$$

$$-\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2}, \qquad \text{odd modes} \qquad (17.2.24)$$

Note that for equation (17.2.1), when we have two parallel metallic plates, $R^{TM} = 1$, and $R^{TE} = -1$, and the guidance condition becomes

$$1 = e^{-2j\beta_{1x}d} \quad \Rightarrow \beta_{1x} = \frac{m\pi}{d}, \qquad m = 0, 1, 2, \ldots, \tag{17.2.25}$$

These are just the guidance conditions for parallel plate waveguides.

17.2.3 A Note on Cut-Off of Dielectric Waveguides

The concept of cut-off in dielectric waveguides is quite different from that of hollow waveguides that we shall learn next. A mode is guided in a dielectric waveguide if the wave is trapped inside, in this case, the dielectric slab. The trapping is due to the total internal reflections at the top and the bottom interfaces of the waveguide. When total internal reflection ceases to occur at any of the two interfaces, the wave is not guided or trapped inside the dielectric slab anymore. This happens when $\alpha_{ix} = 0$ where i can indicate the top-most or the bottom-most region. In other words, the wave ceases to be evanescent in one of the Region i's.